On Cliques and Lagrangians of 3-uniform Hypergraphs

Yuejian Peng * Hegui Zhu † Cheng Zhao ‡

November 29, 2012

Abstract

There is a remarkable connection between the maximum clique number and the Lagrangian of a graph given by T. S. Motzkin and E.G. Straus in 1965. This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique number in graphs. It is useful in practice if similar results hold for hypergraphs. In this paper, we explore evidences that the Lagrangian of a 3-uniform hypergraph is related to the order of its maximum cliques when the number of edges of the hypergraph is in certain range. In particular, we present some results about a conjecture introduced by Y. Peng and C. Zhao (2012) and describe a combinatorial algorithm that can be used to check the validity of the conjecture.

1 Introduction

In 1941, Turán [22] provided an answer to the following question: What is the maximum number of edges in a graph with n vertices not containing a complete subgraph of order l, for a given l? This is the well-known Turán theorem. In 1965, another classical paper by Motzkin and Straus [8] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using the Lagrangian of a graph. This new proof aroused interests in the study of Lagrangians of hypergraphs. Furthermore, the Motzkin-Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem, and the Motzkin-Straus theorem has been also generalized to vertex-weighted graphs [6] and edge-weighted graphs with applications to pattern recognition in image analysis (see [1], [2], [3], [6], [9], [10], [16]). It is useful in practice if similar results hold for hypergraphs. In this paper, we provide evidences that the Lagrangian of an r-uniform hypergraph is related to the order of its maximum cliques under some conditions. We first state a few definitions.

For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2\ldots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $\mathbb{N}$ be the set of all positive integers. For any integer $n \in \mathbb{N}$, denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. We also let $[n]^{(r)}$ represent the complete r-graph on the vertex set $[n]$. When $r = 2$, an r-graph is a simple graph. When $r \geq 3$, an r-graph is often called a hypergraph.

*School of Mathematics, Hunan University, Changsha 410082, P.R. China and Indiana State University, Terre Haute, IN, 47809, USA. Email: ypeng1@163.com

†Mathematics School, Institute of Jilin University, Changchun, 130012, China, and College of Sciences, Northeastern University, Shenyang, 110819, China. Email: zhuhegui@yahoo.com.cn

‡Department of Mathematics and Computer Science, Indiana State University, Terre Haute, IN, 47809 and School of Mathematics, Jilin University, Changchun 130022, P.R. China. Email: cheng.zhao@indstate.edu

Definition 1.1 *Let G be an r-graph with vertex set $[n]$ and edge set $E(G)$. Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) \in R^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, n\}$. For $\vec{x} = (x_1, x_2, \ldots, x_n) \in S$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

The Lagrangian of G, denoted by $\lambda(G)$, is defined as $\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}$. A vector $\vec{y} \in S$ is called an optimal weighting *for G if $\lambda(G, \vec{y}) = \lambda(G)$.*

The following fact is easily implied by the definition of the Lagrangian.

Fact 1.1 *Let G_1, G_2 be r-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

In [8], Motzkin and Straus proved that the Lagrangian of a 2-graph is determined by the order of its maximum clique.

Theorem 1.2 *(Motzkin and Straus [8]) If G is a 2-graph in which a largest clique has order l then $\lambda(G) = \lambda(K_l^{(2)}) = \lambda([l]^{(2)}) = \frac{1}{2}(1 - \frac{1}{l})$.*

The obvious generalization of Motzkin and Straus' result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [17]. Recently, in [14] and [15] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques with applications in image analysis. Although, the obvious generalization of Motzkin and Straus' result to hypergraphs is false, we attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain range. In [12], the following two conjectures are proposed.

Conjecture 1.3 *(Peng-Zhao [12]) Let l and m be positive integers satisfying $\binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1}$. Let G be an r-graph with m edges containing a clique of order $l-1$. Then $\lambda(G) = \lambda([l-1]^{(r)})$.*

Conjecture 1.4 *(Peng-Zhao [12]) Let G be an r-graph with m edges and without containing a clique of order $l-1$, where $\binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1}$. Then $\lambda(G) < \lambda([l-1]^{(r)})$.*

The upper bound $\binom{l-1}{r} + \binom{l-2}{r-1}$ in Conjecture 1.3 is the best possible. For example, if $m = \binom{l-1}{r} + \binom{l-2}{r-1} + 1$ then $\lambda(C_{r,m}) > \lambda([l-1]^{(r)})$, where $C_{r,m}$ is the r-graph on the vertex set $[l]$ and with the edge set $[l-1]^{(r)} \cup \{i_1 \cdots i_{r-1} l, i_1 \cdots i_{r-1} \in [l-2]^{(r-1)}\} \cup \{1 \cdots (r-2)(l-1)l\}$ (note that, take $\vec{x} = (x_1, \ldots, x_l) \in S$, where $x_1 = x_2 = \cdots = x_{l-2} = \frac{1}{l-1}$ and $x_{l-1} = x_l = \frac{1}{2(l-1)}$, then $\lambda(C_{r,m}) \geq \lambda(C_{r,m}, \vec{x}) > \lambda([l-1]^{(r)})$.)

In the course of estimating Turán densities of hypergraphs by applying the Lagrangians of related hypergraphs, Frankl and Füredi [4] asked the following question: Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an r-graph with m edges be? In order to state their conjecture on this problem we require the following definition. For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. Let $C_{r,m}$ denote the r-graph with m edges formed by taking the first m elements in the colex ordering of $\mathbb{N}^{(r)}$.

The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the above question.

Conjecture 1.5 *(Frankl and Füredi [4]) The r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. In particular, the r-graph with $\binom{l}{r}$ edges and the largest Lagrangian is $[l]^{(r)}$.*

This conjecture is true when $r = 2$ by Theorem 1.2. For the case $r = 3$, Talbot in [18] proved the following.

Theorem 1.6 *(Talbot [18]) Let m and l be integers satisfying*

$$\binom{l-1}{3} - 2 \leq m \leq \binom{l-1}{3} + \binom{l-2}{2} - (l-1).$$

Then Conjecture 1.5 is true for $r = 3$ and this value of m.

In [19] and [20], Theorem 1.6 is improved as follows.

Theorem 1.7 *(Tang, Peng, Zhang, and Zhao) Let m and l be integers satisfying*

$$\binom{l-1}{3} - 5 \leq m \leq \binom{l-1}{3} + \binom{l-2}{2} - (l-3).$$

Then Conjecture 1.5 is true for $r = 3$ and this value of m.

Remark 1.8 *The truth of Frankl and Füredi's conjecture is not known in general for $r \geq 4$. Even in the case $r = 3$, the case when $\binom{l}{3} - 6 \leq m \leq \binom{l-1}{3} + \binom{l-2}{2} - (l-4)$ is still open in this conjecture. In [7] and [20], Frankl and Füredi's conjecture for small values m is verified when $r = 3$.*

The following result is given in [18].

Lemma 1.9 *(Talbot [18]) For any integers m, l, and r satisfying $\binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([l-1]^{(r)})$.*

Remark 1.10 *Based on Lemma 1.9, we can see that if both Conjectures 1.3 and 1.4 are true, then Conjecture 1.5 is true for this range of m. It is easy to see that if $\binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1}$, then Conjecture 1.5 implies Conjecture 1.3.*

In [12], it has been shown that Conjecture 1.3 holds when $r = 3$.

Theorem 1.11 *(Peng-Zhao [12]) Let m and l be positive integers satisfying $\binom{l-1}{3} \leq m \leq \binom{l-1}{3} + \binom{l-2}{2}$. Let G be a 3-graph with m edges and contain a clique of order $l-1$. Then $\lambda(G) = \lambda([l-1]^{(3)})$.*

In [11], it has been shown that Conjecture 1.4 holds when $r = 3$ with some conditions.

Theorem 1.12 *(Peng-Tang-Zhao [11]) Let m and l be positive integers satisfying $\binom{l-1}{3} \leq m \leq \binom{l-1}{3} + \binom{l-2}{2} - (l-2)$. Let G be a 3-graph with m edges and without containing a clique of order $l-1$. Then $\lambda(G) < \lambda([l-1]^{(3)})$.*

Further evidences for Conjectures 1.3 and 1.4 are provided in [11] and [21] when the number of vertices of the r-graph is restricted to be a fixed number l.

In this paper, we provide more evidence for Conjecture 1.4 when $r = 3$. In particular, to confirm Conjecture 1.4 when $r = 3$, we prove that we only need to consider 3-graphs with fixed number l vertices, and thus we may describe a combinatorial algorithm of verifying Conjecture 1.4 for given l vertices in Section 3. We point out that a preliminary related partial results was given in [13]. Let us state some useful results in the following section.

2 Preliminary Results

For an r-graph $G = (V, E)$ we denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, we will denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, we will denote the complement of E_{ij} by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$. Denote

$$E_{i \backslash j} = E_i \cap E_j^c.$$

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for an r-graph G:

$$|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \in S \text{ satisfying } |\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \quad (1)$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G)$, subject to $\sum_{i=1}^n x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following lemma gives some necessary condition of an optimal weighting of $\lambda(G)$.

Lemma 2.1 *(Frankl and Rödl [5]) Let $G = (V, E)$ be an r-graph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G with k $(\leq n)$ non-zero weights $x_1, x_2, \dots, x_k$ satisfying condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j.*

Lemma 2.2 *Suppose G is an r-graph on the vertex set $[n]$ with edge set E. Let $1 \leq i < j \leq n$. If $E_{j \backslash i} = \emptyset$, then there exists an optimal weighting $\vec{y} = (y_1, y_2, \dots, y_n)$ of $\lambda(G)$ such that $y_i \geq y_j$.*

Proof. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G. If $x_i < x_j$, then let $y_k = x_k$ for $k \neq i, j$, $y_i = x_j$ and $y_j = x_i$. Then $\vec{y} = (y_1, y_2, \dots, y_n)$ is a weighting for G with $y_i > y_j$ and

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = (x_j - x_i)(\lambda(E_{i \backslash j}, \vec{x}) - \lambda(E_{j \backslash i}, \vec{x})) \geq 0.$$

So $\vec{y}$ is an optimal weighting satisfying the condition. ∎

We call two vertices i, j of an r-graph$G = (V, E)$ equivalent if for all $f \in \binom{V - \{i,j\}}{r-1}$, $f \in E_i$ if and only if $f \in E_j$.

Lemma 2.3 *(c.f. [5]) Suppose G is an r-graph on the vertex set $[n]$. If vertices i_1, i_2, ..., i_t are pairwisely equivalent, then there exists an optimal weighting $\vec{y} = (y_1, y_2, \dots, y_n)$ of $\lambda(G)$ such that $y_{i_1} = y_{i_2} = \dots = y_{i_t}$.*

We say that an r-graph $G = (V, E)$ on the vertex set $[n]$ is *left compressed* if $E_{j \backslash i} = \emptyset$ for any $1 \leq i < j \leq n$. In other words, for any $i < j$, if $k_1 k_2 \dots k_{r-1} \in E_j$, where $k_1, k_2, \dots, k_{r-1} \neq i$, then $k_1 k_2 \dots k_{r-1} \in E_i$.Equivalently, an r-graph $G = (V, E)$ is *left compressed* if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$. We will define *right compressed* similarly later.

Remark 2.4 *(a) In Lemma 2.1, part(a) implies that $x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i \backslash j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j \backslash i}, \vec{x})$. In particular, if G is left compressed, then*

$$(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i \backslash j}, \vec{x}) \quad (2)$$

for any i, j satisfying $1 \leq i < j \leq k$ since $E_{j \backslash i} = \emptyset$.

(b) By (2), if G is left-compressed, then an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for G must satisfy

$$x_1 \geq x_2 \geq \dots \geq x_n \geq 0. \quad (3)$$

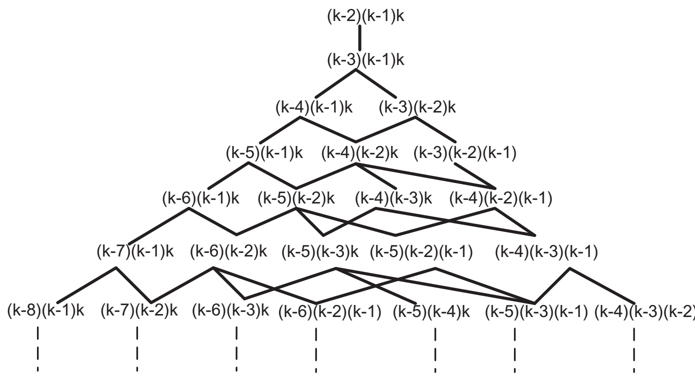

Figure 1: Hessian diagram of triples on k vertices

3 Evidences for Conjecture 1.4

In this section, we describe a procedure that can be used to confirm Conjecture 1.4 when $r = 3$ and l is small. Denote $\lambda^{r-}_{(m,l)} = \max\{\lambda(G) : G$ is an r - graph with m edges and does not contain a clique of size $l\}$.

We need a reduction lemma to simplify our procedure for verification of Conjecture 1.4.

Lemma 3.1 *Let m and l be positive integers satisfying $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$. Then there exists a left compressed 3-graph G on the vertex set $[l]$ with m edges such that $\lambda(G) = \lambda^{3-}_{(m,l-1)}$.*

Remark 3.2 *Lemma 3.1 is used in proving some partial results for confirming Conjecture 1.4 in [11] and [21]. A similar lemma was given in [13], however that proof was flawed.*

To verify Lemma 3.1, we define a partial order Hessian diagram K, a 2-graph on vertices of all triples. A triple $i_1i_2i_3$ is called an *ancestor* of a triple $j_1j_2j_3$ if $i_1 \ge j_1$, $i_2 \ge j_2$, $i_3 \ge j_3$, and $i_1+i_2+i_3 > j_1+j_2+j_3$. In this case, the triple $j_1j_2j_3$ is called a *descendant* of $i_1i_2i_3$. We say that $i_1i_2i_3$ has higher hierarchy than $j_1j_2j_3$ if $i_1i_2i_3$ is an ancestor of $j_1j_2j_3$. A triple $i_1i_2i_3$ is called a *directed ancestor* of a triple $j_1j_2j_3$ if $i_1 \ge j_1$, $i_2 \ge j_2$, $i_3 \ge j_3$, and $i_1 + i_2 + i_3 = j_1 + j_2 + j_3 + 1$. In this case, the triple $j_1j_2j_3$ is called a *directed descendant* of $i_1i_2i_3$, and vertices $i_1i_2i_3$ and $j_1j_2j_3$ are adjacent in the corresponding Hessian diagram. Note that hierarchy is a partial ordering. Figure 1 shows part of the hierarchy relationship of triples in $[k]^{(3)}$. Note that a 3-graph G is left-compressed if and only if for any edge in G, all its descendants (in Hessian diagram K) should be in G as well.

Proof of Lemma 3.1. Let G be a 3-graph on the vertex set $[n]$ with m edges without containing a clique of order $l-1$ such that $\lambda(G) = \lambda^{3-}_{(m,l-1)}$. We call such a 3-graph G an extremal 3-graph for m and $l-1$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting of G. We can assume that $x_i \ge x_j$ when $i < j$ since otherwise we can just relabel the vertices of G and obtain another extremal 3-graph for m and $l-1$ with an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ satisfying $x_i \ge x_j$ when $i < j$. Next we obtain a new 3-graph H from G by performing the following:

1. If $(l-3)(l-2)(l-1) \in E(G)$, then there is at least one triple in $[l-1]^{(3)} \setminus E(G)$, we replace $(l-3)(l-2)(l-1)$ by this triple;

2. If an edge in G has a descendant other than $(l-3)(l-2)(l-1)$ that is not in $E(G)$, then replace this edge by a descendant other than $(l-3)(l-2)(l-1)$ with the lowest hierarchy. Repeat this until there is no such an edge.

Then H satisfies the following properties:

1. The number of edges in H is the same as the number of edges in G.
2. $\lambda(G) = \lambda(G, \vec{x}) \le \lambda(H, \vec{x}) \le \lambda(H)$.
3. $(l-3)(l-2)(l-1) \notin E(H)$.
4. For any edge in $E(H)$, all its descendants other than $(l-3)(l-2)(l-1)$ will be in $E(H)$.

If H is not left-compressed, then there is an ancestor uvw of $(l-3)(l-2)(l-1)$ such that $uvw \in E(H)$. We claim that uvw must be $(l-3)(l-2)l$. If uvw is not $(l-3)(l-2)l$, then since all descendants other than $(l-3)(l-2)(l-1)$ of uvw will be in $E(H)$, then all descendants of $(l-3)(l-1)l$ (other than $(l-3)(l-2)(l-1)$) or all descendants of $(l-3)(l-2)(l+1)$ (other than $(l-3)(l-2)(l-1)$) will be in $E(H)$. So all triples in $[l-1]^{(3)} \setminus \{(l-3)(l-2)(l-1)\}$, all triples in the form of ijl (where $ij \in [l-2]^{(2)}$), and all triples in the form of $ij(l+1)$ (where $ij \in [l-2]^{(2)}$) or all triples in the form of $i(l-1)l$, $1 \le i \le l-3$ will be in $E(H)$, then

$$m \ge \binom{l-1}{3} - 1 + \binom{l-2}{2} + (l-3) > \binom{l-1}{3} + \binom{l-2}{2}$$

which is a contradiction. So uvw must be $(l-3)(l-2)l$. Since $m \le \binom{l-1}{3} + \binom{l-2}{2}$ and all the descendants other than $(l-3)(l-2)(l-1)$ of an edge in H will be an edge in H, then there are two possibilities.

- Case 1. $E(H) = ([l-1]^{(3)} \setminus \{(l-1)(l-2)(l-3)\}) \cup \{ijl, ij \in [l-2]^{(2)}\} \cup \{12(l+1)\}$.
- Case 2. $E(H) = ([l-1]^{(3)} \setminus \{(l-1)(l-2)(l-3)\}) \cup \{ijl, ij \in [l-2]^{(2)}\}$.

Let $\vec{y} = (y_1, y_2, \dots, y_n)$ be an optimal weighting of H, where $n = l+1$ or $n = l$. We claim that if Case 1 happens, then $y_{l+1} = 0$. Notice that $E_{(l+1)\setminus i} = \emptyset$ for each $1 \le i \le l$, by Lemma 2.2, we can assume that $y_i \ge y_{l+1}$ for each $1 \le i \le l$. If $y_{l+1} > 0$, then each $y_i > 0$. This contradicts to $E_{l(l+1)} = \emptyset$ by Lemma 2.1. So we may just consider Case 2.

Since $l-1$, $l-2$ and $l-3$ are pairwisely equivalent in H (Case 2), then by Lemma 2.3, we can assume that $y_{l-1} = y_{l-2} = y_{l-3}$. If $y_{l-1} = y_{l-2} = y_{l-3} = 0$, then clearly $\lambda(H) = \lambda(H, \vec{y}) < \lambda([l-1]^{(3)})$, therefore, $\lambda(G) < \lambda([l-1]^{(3)})$ which confirms Conjecture 1.4. So assume that $y_{l-1} = y_{l-2} = y_{l-3} > 0$. Let F be the 3-graph with edge set

$$E(F) = ([l-1]^{(3)} \cup \{ijl, ij \in [l-2]^{(2)}\}.$$

Then

$$\lambda(G) \le \lambda(H) = \lambda(H, \vec{y}) < \lambda(F, \vec{y}) \le \lambda(F) = \lambda([l-1]^{(3)})$$

which confirms Conjecture 1.4. So we can assume that H is left-compressed. Clearly H does not contain $[l-1]^{(3)}$. Since H is left-compressed, then H does not contain a clique of order $l-1$. Therefore we get a left-compressed extremal 3-graph H for m and $l-1$. Hence we can assume that G is left -compressed.

Next show that we can assume G is on l vertices. Let k be the number of positive coordinates in $\vec{x}$ (optimal weighting for G). If $k \le l-1$, then $\lambda(G) \le \lambda([l-1]^{(3)} \setminus \{(l-1)(l-2)(l-3)\}) < \lambda([l-1]^{(3)})$ which confirms Conjecture 1.4. So we may assume that $k \ge l$. We will use Lemma 3.3 below. The proof of Lemma 3.3 is similar to a proof of a result in [18]. However, the result in [18] cannot be applied directly here. For completeness, the proof of Lemma 3.3 will be given in the appendix.

Lemma 3.3 *Let m and l be positive integers satisfying $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$. Let G be a left-compressed 3-graph on the vertex set $[k]$ with m edges and without containing a clique of order $l-1$ such that $\lambda(G) = \lambda^{3-}_{(m,l-1)}$. Let $\vec{x}$ be an optimal weighting for G with k positive weights. Then Conjecture 1.4 holds or*

$$|[k-1]^{(3)} \setminus E| \le k-2.$$

Now assume that Lemma 3.3 holds and continue the proof of Lemma 3.1. Since G is left compressed, then $1(k-1)k \in E$ and $|[k-2]^{(2)} \cap E_k| \geq 1$.

If $k \geq l+1$, then applying Lemma 3.3, we have

$$\begin{aligned} m = |E| &= |E \cap [k-1]^{(3)}| + |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \\ &\geq \binom{l}{3} - (l-1) + 2 \\ &\geq \binom{l-1}{3} + \binom{l-2}{2} + 1, \end{aligned}$$

which contradicts to the assumption that $m \leq \binom{l-1}{3} + \binom{l-2}{2}$. Recall that $k \geq l$, so we have $k = l$. Since $\vec{x}$ has only l positive weights, we can assume that G is on l vertices. This proves Lemma 3.1. ■

Theorem 3.4 *To verify Conjecture 1.4 for $r = 3$ and any given l, it is sufficient to verify $\lambda(G) < \lambda([l-1]^{(3)})$ for all left-compressed 3-graphs G on the vertex set $[l]$ with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$.*

Proof: Since $\lambda^{3-}_{(m,l-1)}$ does not decrease as m increases, it is sufficient to verify Conjecture 1.4 for $m = \binom{l-1}{3} + \binom{l-2}{2}$. By Lemma 3.1, it is sufficient to verify $\lambda(G) < \lambda([l-1]^{(3)})$ for all left compressed 3-graphs G on l vertices with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$.

Now we describe an Algorithm to produce all left compressed 3-graphs G on the vertex set $[l]$ with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$. Notice that, for a 3-graph G on l vertices with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing a clique of order $l-1$, we may write $[l]^{(3)}$ as

$$[l]^{(3)} = [l-1]^{(3)} \cup G_1 \cup G_2$$

where $G_1 = \{ijl, \text{where } ij \in [l-2]^{(2)}\}$ and $G_2 = \{1(l-1)l, 2(l-1)l, 3(l-1)l, \cdots, (l-2)(l-1)l\}$.

Under these assumptions, G can be obtained from $[l]^{(3)}$ by deletion of a subgraph H with $l-2$ edges since $l-2 = \binom{l}{3} - [\binom{l-1}{3} + \binom{l-2}{2}]$. These $l-2$ edges of H consist of edges from $[l-1]^{(3)}$, G_1 (if any), or G_2. Specifically, 3-graph G takes the form of

$$G = ([l-1]^{(3)} - E_1) \cup (G_1 - E_2) \cup \{1(l-1)l, 2(l-1)l, 3(l-1)l, \cdots, i(l-1)l\}$$

for some edge set E_1 from $[l-1]^{(3)}$, some edge set E_2 from G_1, and some i where $1 \leq i \leq l-3$. Observe that $E_{q \setminus j} = \emptyset$ for $1 \leq q < j \leq i$. By (2), $x_1 = x_2 = \cdots = x_i$.

An r-graph $H = (V, E)$ on the vertex set $[l]$ is right-compressed if $i_1 i_2 i_3 \in E$ implies $j_1 j_2 j_3 \in E$ whenever $j_1 \geq i_1$, $j_2 \geq i_2$, and $j_3 \geq i_3$. Note that H is right-compressed if and only if the complement of H is left-compressed. To generate all possible left compressed 3-graphs on the vertex set $[l]$ with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$, we can generate all right compressed connected 3-subgraphs H rooted at $(l-2)(l-1)l$ with $l-2$ edges and containing $(l-3)(l-2)(l-1)$ and then take the complement of each H. To do so, we use Algorithm 3.5 in the following procedure that is based on Figure 1 (replacing k by l).

Algorithm 3.5 *List all left compressed 3-graphs on the vertex set $[l]$ with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$.*

Input: $l \geq 7$ and a Hessian graph with vertex set $[l]^{(3)}$ (replace k by l in Figure 1).

Output: All right compressed connected 3-subgraphs H rooted at $(l-2)(l-1)l$ with $l-2$ edges and containing $(l-3)(l-2)(l-1)$, thus produce all possible left compressed 3-graph $G = [l]^{(3)} - H$ on the vertex set $[l]$ with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges and without containing the clique $[l-1]^{(3)}$.

Initialization: Set $H = \{(l-2)(l-1)l, (l-3)(l-1)l, (l-3)(l-2)l, (l-3)(l-2)(l-1), (l-4)(l-1)l\}$.

Step 1. For each direct descendant of H (an edge e is a direct descendant of H if $e \in E(H)$ and e is a direct descendant of an edge in H), check whether all its direct ancestors are in H. If so, add to H.

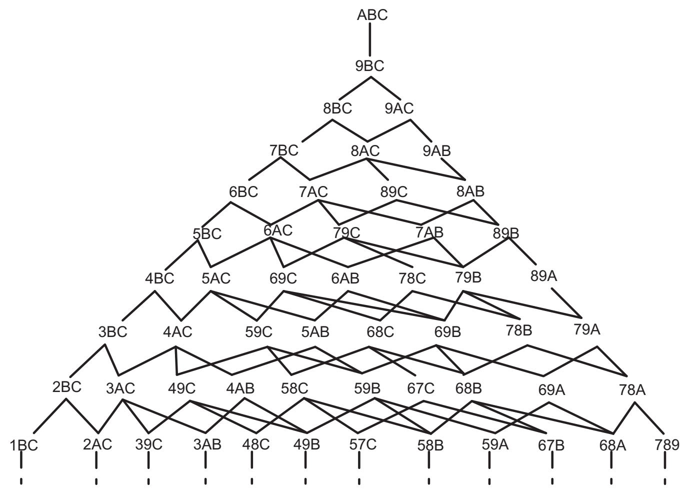

Figure 2: Hessian diagram of triples on 12 vertices

Then record the new H and record the size of the new H. Take all distinct new H with size increased by 1 and repeat this process until there are $l-2$ triples in H. Output all distinct H with $l-2$ triples.

Step 2. Performing $G = [l]^{(3)} - H$ and output G.

If $l = 6$, then $G = [6]^{(3)} - H$ is the only left-compressed 3-graph on $[6]$ with $\binom{5}{3} + \binom{4}{2}$ edges without containing a clique of order 5, where $H = \{456, 356, 346, 345\}$.

It is obvious that Algorithm 3.5 leads to the following result.

Lemma 3.6 *Algorithm 3.5 produces all possible left compressed 3-graphs on $l \geq 7$ vertices with $m = \binom{l-1}{3} + \binom{l-2}{2}$ edges.*

As a demonstration, we verify Conjecture 1.4 when l is small in Proposition 3.7.

Proposition 3.7 *Let G be a 3-graph on l vertices with m edges and contain no clique of order $l-1$, where $m = \binom{l-1}{3} + \binom{l-2}{2}$. Then $\lambda(G) < \lambda([l-1]^{(3)})$ for $6 \leq l \leq 12$.*

The detailed proof of the case $l = 12$ will be given in the next section.

4 Proof of Proposition 3.7

We may verify the proposition by case analysis for each given l. Here we just demonstrate the case when $l = 12$. Using Algorithm 3.5, we list all left compressed 3-graphs on vertex set $\{1, 2, \cdots, 12\}$ with $m = 220 = \binom{12-1}{3} + \binom{12-2}{2}$ edges without containing a clique of order 11. Let us write $A = 10$, $B = 11$, and $C = 12$ in Figure 2.

Now applying Algorithm 3.5, start from initialization: let $H = \{ABC, 9BC, 9AC, 9AB, 8BC\}$. In step 1, the direct descendants of H are $7BC$, $8AC$, and $8AB$. Note that all ancestors of $7BC$ and all ancestors of $8AC$ are in H. We could add $7BC$ to H or add $8AC$ to H. So there are two new H with size increased by 1: $H = \{ABC, 9BC, 9AC, 9AB, 8BC, 7BC\}$ or $H = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC\}$. For each of them, repeat the procedure. Continue this process until there are 10 edges in the updated H. Output all distinct H with 10 edges:

- $H_1 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 89C, 7BC, 7AC, 6BC\}$,

- $H_2 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 7BC, 7AC, 6BC\}$,
- $H_3 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 89C, 7BC, 6BC\}$,
- $H_4 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 89C, 7BC, 7AC\}$,
- $H_5 = \{ABC, 9BC, 9AC, 9AB, 8BC, 7BC, 6BC, 5BC, 4BC, 3BC\}$,
- $H_6 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 7BC, 6BC, 5BC, 4BC\}$,
- $H_7 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 7BC, 7AC, 6BC, 5BC\}$,
- $H_8 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 89C, 7BC, 6BC, 5BC\}$,
- $H_9 = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 7BC, 6BC, 5BC\}$,
- $H_{10} = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 89C, 89B, 89A\}$,
- $H_{11} = \{ABC, 9BC, 9AC, 9AB, 8BC, 8AC, 8AB, 7BC, 7AC, 7AB\}$.

Correspondingly, by Step 2, we may list 11 left compressed 3-graphs on the vertex set $[l]$ where $l = 12$ with 220 edges.

Now we may verify Conjecture 1.4 by direct calculation of the corresponding Lagrangian values from these eleven 3-graphs. We divide our case analysis into subcases.

Subcase 1. The set H_1 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 7A, 89\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC\}$. Observe that $E_{i\setminus j} = \emptyset$ for$1 \leq i < j \leq 5$. By (2), $x_1 = x_2 = x_3 = x_4 = x_5 \overset{\text{def}}{=} a$. Let $x_6 \overset{\text{def}}{=} b$, $x_7 \overset{\text{def}}{=} c$, $x_8 \overset{\text{def}}{=} d$, $x_9 \overset{\text{def}}{=} e$, $x_{10} \overset{\text{def}}{=} f$, $x_{11} \overset{\text{def}}{=} g$, $x_{12} \overset{\text{def}}{=} h$. Note that $5a + b + c + d + e + f + g + h = 1$. In viewing of edges in E,

$$\begin{aligned}
\lambda(G) &= \binom{5}{3}a^3 + \binom{5}{2}a^2(b + c + d + e + f + g) \\
&\quad + 5a(bc + bd + be + bf + bg + cd + ce + cf + cg + de + df + dg + ef + eg + fg) \\
&\quad + (bcd + bce + bcf + bcg) + (bde + bdf + bdg) \\
&\quad + (bef + beg + bfg) + (cde + cdf + cdg) \\
&\quad + (cef + ceg + cfg) + (def + deg + dfg) \\
&\quad + h[\binom{5}{2}a^2 + 5a(b + c + d + e + f) + bc + bd + be + bf + cd + ce] + 5agh \\
&\overset{\text{def}}{=} G_1(a, b, c, d, e, f, g, h).
\end{aligned}$$

Using the software Matlab, testing result shows that the maximum value of $G_1(a, b, c, d, e, f, g)$ under the constraint

$$5a + b + c + d + e + f + g + h = 1, a \geq b \geq c \geq d \geq e \geq f \geq g \geq h \geq 0.$$

is $\leq 0.1211650204992799 < \lambda([11]^3) = \frac{15}{121} = 0.1240$.

For the rest of 10 subcases, we only summarize our computation results as follows.

Subcase 2. The set H_2 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 7A\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC\}$. Using the software Matlab, testing result shows that $\lambda(G) \leq 12156108878284182 < \lambda([11]^3) = \frac{15}{121} = 0.1240$ in this subcase.

Subcase 3. The set H_3 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 89\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC\}$. Matlab testing result shows that $\lambda(G)$ is $\leq 0.1230233663938971 < \frac{15}{121} = 0.1240$ in this subcase.

Subcase 4. The set H_4 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 7A, 89\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC, 6BC\}$. Matlab testing result shows that $\lambda(G)$ is $\leq 0.12181063749855171 < \frac{15}{121} = 0.1240$ in this subcase.

Subcase 5. The set H_5 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A\}\} \cup \{1BC, 2BC\}$. Matlab testing result shows that $\lambda(G)$ is $\leq 0.12240632785698001 < \frac{15}{121} = 0.1240$ in this subcase.

Subcase 6. The set H_6 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A\}\} \cup \{1BC, 2BC, 3BC\}$. Computation shows that $\lambda(G)$ is $\leq 0.12187845719377105 < \frac{15}{121} = 0.1240$.

Subcase 7. The set H_7 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 7A\}\} \cup \{1BC, 2BC, 3BC, 4BC\}$. Matlab computation testing shows that $\lambda(G)$ is $\leq 0.12320953232482997 < \frac{15}{121} = 0.1240$.

Subcase 8. The set H_8 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 89\}\} \cup \{1BC, 2BC, 3BC, 4BC\}$. Matlab computation testing shows that $\lambda(G)$ is $\leq 0.12205596090717179 < \frac{15}{121} = 0.1240$.

Subcase 9. The set H_9 is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A\}\} \cup \{1BC, 2BC, 3BC, 4BC\}$. Matlab computation testing shows that $\lambda(G)$ is $\leq 0.12141031243761445 < \frac{15}{121} = 0.1240$.

Subcase 10. The set H_{10} is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB, 89B, 89A\}) \cup \{Cij, ij \in [10]^{(2)} \setminus \{9A, 8A, 89\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC, 6BC, 7BC\}$. Testing result shows that $\lambda(G)$ is $\leq 0.12140780421728305 < \frac{15}{121} = 0.1240$.

Subcase 11. The set H_{11} is removed from $[12]^{(3)}$. In this case, $E = ([11]^{(3)} \setminus \{9AB, 8AB, 7AB\}) \cup \{12ij, ij \in [10]^{(2)} \setminus \{9A, 8A, 7A\}\} \cup \{1BC, 2BC, 3BC, 4BC, 5BC, 6BC\}$. Testing result shows that $\lambda(G)$ is $\leq 0.12148307198540859 < \frac{15}{121} = 0.1240$.

Now the verification of Proposition 3.7 is completed. ■

References

[1] I. M. Bomze, Evolution towards the maximum clique. *Journal of Global Optimization*, 10(2):143-164, 1997.

[2] M. Budinich, Exact bounds on the order of the maximum clique of a graph. *Discrete Applied Mathematics*, 127:535-543, 2003.

[3] S. Busygin, A new trust region technique for the maximum weight clique problem. *Discrete Applied Mathematics*, 304(4):2080-2096, 2006.

[4] P. Frankl and Z. Füredi, Extremal problems whose solutions are the blow-ups of the small witt-designs. *Journal of Combinatorial Theory (A)*, 52:129-147, 1989.

[5] P. Frankl and V. Rödl, Hypergraphs do not jump. *Combinatorica*, 4:149-159, 1984.

[6] L. E. Gibbons, D. W. Hearn, P. M. Pardalos, and M. V. Ramana, Continuous characterizations of the maximum clique problem. *Mathematics of Operations Research*, 22: 754-768, 1997.

[7] G. He, Y. Peng, and C. Zhao, Finding lagrangians of hypergraphs. *Ars Combinatoria* (accepted).

[8] T. S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Turán. *Canadian Journal of Mathematics*, 17:533-540, 1965.

[9] M. Pavan and M. Pelillo, Generalizing the Motzkin-Straus theorem to edge-weighted graphs, with applications to image segmentation. *Lecture Notes in Computer Science*, 2683:485-500, 2003.

[10] P. M. Pardalos and A.T. Phillips, A global optimization approach for solving the maximum clique problem. *International Journal of Computer Mathematics*, 33:209-216, 1990.

[11] Y. Peng, Q. S. Tang, and C. Zhao, On lagrangians of r-uniform hypergraphs, submitted.

[12] Y. Peng and C. Zhao, A Motzkin-Straus type result for 3-uniform hypergraphs, *Graphs and Combinatorcs*, accepted, online: DOI 10.1007/s00373-012-1135-5.

[13] Y. Peng and C. Zhao, On Lagrangians of hypergraphs and cliques. In *Recent Advances in Computer Science and Information Engineering*, volume 125, pages 7-12, 2012.

[14] S. R. Buló and M. Pelillo, A continuous characterization of maximal cliques in k-uniform hypergraphs. In *Learning and Intelligent Optimization*, volume 5313, pages 220-233, 2008.

[15] S. R. Buló and M. Pelillo, A generalization of the Motzkin-Straus theorem to hypergraphs. *Optimization Letters*, 3(2): 287-295, 2009.

[16] S.R. Buló, A. Torsello, and M. Pelillo, A continuous-based approach for partial clique enumeration. *Graph-Based Representations in Pattern Recognition*, 4538:61-70, 2007.

[17] V. Sós and E. G. Straus, Extremal of functions on graphs with applications to graphs and hypergraphs. *Journal of Combinatorial Theory, Series B*, 63:189-207, 1982 .

[18] J. Talbot, Lagrangians of hypergraphs, *Combinatorics, Probability & Computing*, 11:199-216, 2002.

[19] Q. S. Tang, Y. Peng, X. D. Zhang, and C. Zhao, Some results on lagrangians of hypergraphs, submitted.

[20] Q. S. Tang, Y. Peng, X. D. Zhang, and C. Zhao, On Frankl and Füredi's conjecture for 3-uniform hypergraphs, submitted.

[21] Q. S. Tang, Y. Peng, X. D. Zhang, and C. Zhao, On Lagrangians of hypergraphs with a specified subgraph, in preparation.

[22] P. Turán, On an extremal problem in graph theory(in Hungarian), *Mat. Fiz. Lapok.*, 48:436-452, 1941.

Appendix: Proof of Lemma 3.3

Let m and l be positive integers satisfying $\binom{l-1}{3} \le m \le \binom{l-1}{3} + \binom{l-2}{2}$. Let G be a left-compressed 3-graph on the vertex set $[k]$ with m edges and without containing a clique of order $l-1$ such that $\lambda(G) = \lambda^{3-}_{(m,l-1)}$. Let $\vec{x}$ be an optimal weighting for G with k positive weights. We show that Conjecture 1.4 holds or

$$|[k-1]^{(3)} \setminus E| \le k-2.$$

Since G is left-compressed, $1(k-1)k \in E$. Let $b = \max\{i : i(k-1)k \in E\}$. Since E is left-compressed, then $E_i = \{1, \dots, i-1, i+1, \dots, k\}^{(2)}$, for $1 \le i \le b$, and $E_{i \setminus j} = \emptyset$ for $1 \le i < j \le b$. Hence, by Remark 2.4(a), we have $x_1 = x_2 = \cdots = x_b$.

We define a new legal weighting $\vec{y}$ for G as follows. Let $y_i = x_i$ for $i \ne k-1, k$, $y_{k-1} = x_{k-1} + x_k$ and $y_k = 0$.

By Lemma 2.1(a), $\lambda(E_{k-1}, \vec{x}) = \lambda(E_k, \vec{x})$, so

$$\begin{aligned}
\lambda(G, \vec{y}) - \lambda(G, \vec{x}) &= x_k(\lambda(E_{k-1}, \vec{x}) - x_k \lambda(E_{k(k-1)}, \vec{x})) \\
&\quad - x_k(\lambda(E_k, \vec{x}) - x_{k-1}\lambda(E_{k(k-1)}, \vec{x})) - x_{k-1}x_k\lambda(E_{k(k-1)}, \vec{x}) \\
&= x_k(\lambda(E_{k-1}, \vec{x}) - \lambda(E_k, \vec{x})) - x_k^2 \sum_{i=1}^{b} x_i \\
&= -b x_1 x_k^2. \qquad (4)
\end{aligned}$$

Since $y_k = 0$ we may remove all edges containing k from E to form a new 3-graph $\bar{G} = ([k], \bar{E})$ with $\lambda(\bar{G}, \vec{y}) = \lambda(G, \vec{y})$ and $|\bar{E}| = |E| - |E_k|$. We will show that if Lemma 3.3 fails to hold then there exists a set of edges $F' \subset [k-1]^{(3)} \setminus E$ satisfying

$$\lambda(F', \vec{y}) > b x_1 x_k^2 \qquad (5)$$

and

$$|F'| \le |E_k|. \qquad (6)$$

Then, using (4), (5), and (6), the 3-graph $G'' = ([k], E'')$, where $E'' = \bar{E} \cup F'$, satisfies $|E''| \le |E|$ and

$$\begin{aligned}
\lambda(G'', \vec{y}) &= \lambda(\bar{G}, \vec{y}) + \lambda(F', \vec{y}) \\
&> \lambda(G, \vec{y}) + b x_1 x_k^2 \\
&= \lambda(G, \vec{x}).
\end{aligned}$$

Hence $\lambda(G'') > \lambda(G)$. If G'' does not contains a clique of size $l-1$, then it contradicts to $\lambda(G) = \lambda^{3-}_{(m,l-1)}$. If G'' contains a clique of size $l-1$, then by Theorem 1.12, $\lambda(G'') = \lambda([l-1]^{(3)})$ and consequently $\lambda(G) < \lambda([l-1]^{(3)})$ which confirms Conjecture 1.4.

We must now construct the set of edges F' satisfying (5) and (6). Applying Remark 2.4(a) by taking $i = 1, j = k-1$, we have

$$x_1 = x_{k-1} + \frac{\lambda(E_{1\setminus(k-1)}, \vec{x})}{\lambda(E_{1(k-1)}, \vec{x})}.$$

Let $C = [k-2]^{(2)} \setminus E_{k-1}$. Then $\lambda(E_{1\setminus(k-1)}, \vec{x}) \le x_k \sum_{i=b+1}^{k-2} x_i + \lambda(C, \vec{x})$. Applying this and multiplying $b x_k^2$ to the above equation (note that $\lambda(E_{1(k-1)}, \vec{x}) = \sum_{i=2, i\ne k-1}^{k} x_i$), we have

$$b x_1 x_k^2 \le b x_{k-1} x_k^2 + \frac{b x_k^3 \sum_{i=b+1}^{k-2} x_i}{\sum_{i=2, i\ne k-1}^{k} x_i} + \frac{b x_k^2 \lambda(C, \vec{x})}{\sum_{i=2, i\ne k-1}^{k} x_i}.$$

Since $x_1 \geq x_2 \geq \cdots \geq x_k$, then

$$bx_1x_k^2 \leq bx_{k-1}x_k^2(1 + \frac{k-(b+2)}{k-3}) + \frac{bx_k\lambda(C,\vec{x})}{k-2}. \tag{7}$$

Define $\alpha = \lceil \frac{b|C|}{k-2} \rceil$ and $\beta = \lceil b(1 + \frac{k-(b+2)}{k-3}) \rceil$. Notice that $\lceil b(1 + \frac{k-(b+2)}{k-3}) \rceil \leq k-2$ since $b \leq k-2$. So $\beta \leq k-2$. Let the set $F_1 \subset [k-1]^{(3)} \setminus E$ consist of the α heaviest edges in $[k-1]^{(3)} \setminus E$ containing the vertex $k-1$ (note that $|[k-2]^{(2)} \setminus E_{k-1}| = |C| \geq \alpha$). Recalling that $y_{k-1} = x_{k-1} + x_k$ we have

$$\lambda(F_1, \vec{y}) \geq \frac{bx_k\lambda(C,\vec{x})}{k-2} + \alpha x_{k-1}x_k^2.$$

So using (7)

$$\lambda(F_1, \vec{y}) - bx_1x_k^2 \geq x_{k-1}x_k^2(\alpha - \beta). \tag{8}$$

We now distinguish two cases.

Case 1. $\alpha > \beta$.

In this case $\lambda(F_1, \vec{y}) - bx_{k-1}x_k^2 > 0$ so defining $F' = F_1$ satisfies (5). We need to check that $|F'| \leq |E_k|$. Since E is left compressed, then $[b]^{(2)} \cup \{1, \ldots, b\} \times \{b+1, \ldots, k-1\} \subset E_k$. Hence

$$|E_k| \geq \frac{b[b-1+2(k-1-b)]}{2} \geq \frac{b(k-1)}{2} \tag{9}$$

since $b \leq k-2$. Recall that $|F'| = \alpha = \lceil \frac{b|C|}{k-2} \rceil$. Since $C \subset [k-2]^{(2)}$, we have $|C| \leq \binom{k-2}{2}$. So using (9) we obtain

$$|F'| \leq \lceil \frac{b(k-3)}{2} \rceil \leq \frac{b(k-1)}{2} \leq |E_k|.$$

So both (5) and (6) are satisfied.

Case 2. $\alpha \leq \beta$.

Suppose that Lemma 3.3 fails to hold. So $|[k-1]^{(3)} \setminus E| \geq k-1 \geq \beta+1$ (recall that $\beta \leq k-2$). Let F_2 consist of any $\beta + 1 - \alpha$ edges in $[k-1]^{(3)} \setminus (E \cup F_1)$ and define $F' = F_1 \cup F_2$. Then since $\lambda(F_2, \vec{y}) \geq (\beta + 1 - \alpha)x_{k-1}^3$ and using (8),

$$\lambda(F', \vec{y}) - bx_{k-1}x_k^2 = \lambda(F_1, \vec{y}) - bx_{k-1}x_k^2 + \lambda(F_2, \vec{y}) \geq (\beta + 1 - \alpha)x_{k-1}^3 - x_{k-1}x_k^2(\beta - \alpha) > 0.$$

So (5) is satisfied. What remains is to check that $|F'| \leq |E_k|$. In fact,

$$|F'| = \beta + 1 \leq k - 1 \leq \frac{b(k-1)}{2} \leq |E_k|$$

when $b \geq 2$. If $b = 1$, then applying (9),

$$|F'| = \beta + 1 = 3 \leq k - 2 = \frac{b[b-1+2(k-1-b)]}{2} \leq |E_k|$$

since $k \geq l \geq 5$. ■